Exercise 7

Evaluate the line integral, where C is the given curve.

$\int_C (x+2y)\,dx + x^2\,dy$, $\quad C$ consists of line segments from $(0,0)$ to $(2,1)$ and from $(2,1)$ to $(3,0)$

Solution

The equation of the line going through $(0,0)$ and $(2,1)$ is

$$y = \frac{1}{2}x,$$

and the equation of the line going through $(2,1)$ to $(3,0)$ is

$$y = -x + 3.$$

Rewrite the given integral using the dot product.

$$\int_C (x+2y)\,dx + x^2\,dy = \int_C \langle x+2y, x^2\rangle \cdot \langle dx, dy\rangle$$

To evaluate it, split it up over the two lines.

$$\int_C (x+2y)\,dx + x^2\,dy = \int_{\text{Line 1}} \langle x+2y, x^2\rangle \cdot \langle dx, dy\rangle + \int_{\text{Line 2}} \langle x+2y, x^2\rangle \cdot \langle dx, dy\rangle$$

Parameterize the first line by $x = t$, which then means $y = t/2$, with $0 \le t \le 2$. Parameterize the second line by $x = t$, which then means $y = -t + 3$, with $2 \le t \le 3$.

$$\begin{aligned}
\int_C (x+2y)\,dx + x^2\,dy &= \int_0^2 \langle x(t) + 2y(t), [x(t)]^2\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt \\
&\qquad + \int_2^3 \langle x(t) + 2y(t), [x(t)]^2\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt \\
&= \int_0^2 \left\langle t + 2\left(\frac{t}{2}\right), t^2\right\rangle \cdot \left\langle 1, \frac{1}{2}\right\rangle dt \\
&\qquad + \int_2^3 \left\langle t + 2(-t+3), t^2\right\rangle \cdot \langle 1, -1\rangle\, dt \\
&= \int_0^2 \langle 2t, t^2\rangle \cdot \left\langle 1, \frac{1}{2}\right\rangle dt + \int_2^3 \langle 6-t, t^2\rangle \cdot \langle 1, -1\rangle\, dt \\
&= \int_0^2 \left(2t + \frac{t^2}{2}\right) dt + \int_2^3 [(6-t) - t^2]\, dt \\
&= \left(\frac{16}{3}\right) + \left(-\frac{17}{6}\right) \\
&= \frac{5}{2}
\end{aligned}$$